On Estimation of the Parameters of Gielis Superformula from Empirical Data

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Introduction: Nature presents abundant examples of closed curves that may be considered as some sort of deformed ellipses. Gielis (2003) invented a superformula to parameterize such shapes.

The classical formula of ellipse is given as:

$$r(\theta)=\left[\left(\frac{\cos(\theta)}{a}\right)^2+\left(\frac{\sin(\theta)}{b}\right)^2\right]^{(-1/2)} \quad ;a,b>0 \qquad \ldots (1)$$

that may, without any loss to generality, be expressed as:

$$r(\theta)=\left[\left|\frac{\cos(\frac{4\theta}{4})}{a}\right|^2+\left|\frac{\sin(\frac{4\theta}{4})}{b}\right|^2\right]^{(-1/2)} \quad ;a,b>0 \qquad \ldots (2)$$

or

$$r(\theta)=\left[\left|\frac{\cos(\frac{m\theta}{4})}{a}\right|^{n_2}+\left|\frac{\sin(\frac{m\theta}{4})}{b}\right|^{n_3}\right]^{(-1/n_1)} \quad ;a,b>0;\ \ m=4; n_1=n_2=n_3=2 \qquad \ldots (3)$$

Thus, we have the formula

$$r(\theta)=f(\theta).\left[\left|\frac{\cos(\frac{m\theta}{4})}{a}\right|^{n_2}+\left|\frac{\sin(\frac{m\theta}{4})}{b}\right|^{n_3}\right]^{(-1/n_1)}=f(\theta).g(\theta)\ ;\ a,b,m>0 \qquad \ldots (4)$$

Gielis found that almost any (closed) curve can be described in terms of $f(\theta)$, which is any defined function of θ, and the parameters $(a,b,m,n_{j=1,2,3})$ in (4) and called it a superformula.

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Let us look at the formulas (1) to (4) above a little differently. The parameters a and b relate to maximal length (along x axis) and width (along y axis). When x and y are divided by a and b respectively, the length and the width are unitized such that all the points lie on or inside the unit circle. If the parameters satisfy the relationship $(n_1 = n_2 = n_3) \geq 1$, we may view (4) above as some sort of 'ball' with a well-defined Minkowski distance (of a point from the origin). The Minkowski distance is measured as

$$d = \left[|u|^p + |v|^p \right]^{1/p} ; p \geq 1 \qquad \ldots (5)$$

Note that this distance satisfies the triangle inequality condition for any real $p \geq 1$, (integer or non-integer). For u and v within the unit ball, p larger than unity would deflate them and $p<1$ would inflate them (violating the Δ inequality). The exponent $(1/p)$ on the square bracket operating on the sum of rescaled u and v (within the square bracket) would have the role opposite to p. However, in the formula

$$d = \left[|u|^p + |v|^q \right]^{1/w} ; (p \neq q \neq, w) \geq 1 \qquad \ldots (6)$$

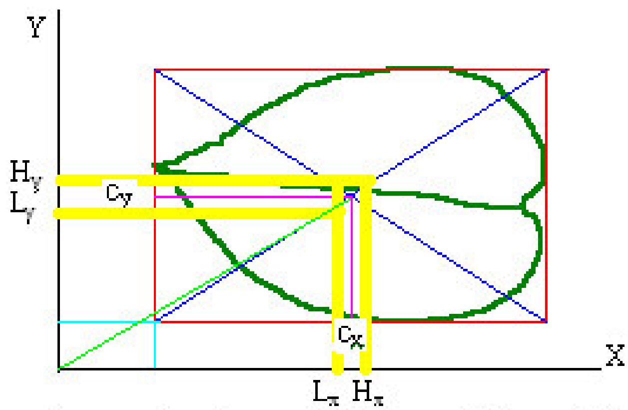
scaling of u and v within the square bracket are not uniform, nor the rescaling of their sum by the exponent on the square bracket has any particular relation with them. They stretch and squeeze the ball in different proportions in different directions.

Thus, the matters are entirely different when $p, q \; or \; w$ assume a negative or a fractional value. Then d is no longer a measure of distance in the sense described above. Gielis' formula allows for such negative or fractional values of $p, q \; or \; w$.

Examples in Nature: Among the plant leaves we find different shapes - linear, lanceolate, elliptic, oblong, ovate, abovate, orbicular, cordate, reniform, sagittate, hastate, auricled, etc. Leaves exhibit different patterns in their margin – entire, undulate, dentate, serrate, crenate, pinnately lobed and so on. Leaf tips may be acuminate, acute, obtuse, truncate, retuse, cuspidate, mucronate, and may others. These particulars characterize plant leaves and the plants in turn. Very often, therefore, plant leaves are qualitatively described. However, Gielis holds that it is possible to describe these shapes in terms of parameters $(a, b, m, n_{j=1,2,3})$.

Measurements: The figure (for instance, a plant leaf) may be placed on a graph paper and traced to give a 2d graph, which may be converted into data points $z_i = (x_i, y_i);\ i = 1, 2, ..., n.$ In practice, these measurements are such that the entire leaf is traced in 4 quadrants, with the centroid placed (approximately) on the origin $(0,0)$ of the graph. Since it is almost impossible in practice to exactly place the centroid of a leaf on the correct origin, (0,0) on a graph paper, we assume that the origin is wrongly located at (c_x, c_y). Accordingly, every point on the trace of the leaf is the distance, r', measured with the origin (c_x, c_y) on the graph, with the coordinates $(x', y') = (x + c_x, y + c_y)$. After obtaining such data, what remains is to fit the Gielis superformula in the data and obtain the estimates of parameters $(\hat{a}, \hat{b}, \hat{m}, \hat{n}_{j=1,2,3})$. This paper is an attempt to explore this possibility.

The First Step to Estimation of Gielis Parameters: The first and most important task in estimation of the Gielis parameters from such empirical data (with displaced origin) is to estimate the shift parameters, (c_x, c_y), so that $(x, y) = (x - c_x, y - c_y)$ can be obtained as correctly as possible. We draw the practically smallest possible rectangle such that the trace of the figure is inscribed within the said rectangle, we may obtain the first estimate of (c_x, c_y) along with the limits within which each shift parameter would lie. That is: $L_x \le c_x \le H_x$; $L_y \le c_y \le H_y$. We can also obtain approximate values of a = half of the length and b =half of the width of the rectangle, and the limits within which they lie such that $L_a \le a \le H_a$; $L_b \le b \le H_b$. *Any other method may also be used to guess them.*



Approximation of Values of C_x and C_y

Having obtained the approximate values of the first four parameters, (c_x, c_y) and (a, b), searching for the values of the rest four parameters (m, n_1, n_2, n_3) needs a guess work and a non-linear search algorithm.

Estimation of Gielis Parameters: Let the n true points be $[z_i = (x_i, y_i);\ i = 1, 2, ..., n]$, of which the corresponding observed values are $z' = (x'_i, y'_i)$, possibly with errors of measurement and displacement of origin by (c_x, c_y), unknown to us. Let $(\tilde{c}_x, \tilde{c}_y)$ be the approximate or assumed values of (c_x, c_y). Let us denote by $\tilde{z}_i = (\tilde{x}_i, \tilde{y}_i) = (x'_i - \tilde{c}_x, y'_i - \tilde{c}_y)$. From these values we obtain $\tilde{r}_i = \sqrt{(\tilde{x}_i^2 + \tilde{y}_i^2)}$. We also obtain $\tilde{\theta}_i = \tan^{-1}(\tilde{y}_i / \tilde{x}_i)$. On the other hand, we obtain $\hat{r}_i = g(\tilde{\theta}_i, \tilde{a}, \tilde{b}, \tilde{m}, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$, where $g(.)$ is the Gielis superformula defined in (4). Presently, we assume $f(\theta) = 1$. The wavy bar on the arguments of $g(.)$ indicates that all parameters have taken on some assumed values, which may not be the correct values. The deviation of assumed values of parameters from their true values gives rise to $d_i = abs(\tilde{r}_i - \hat{r}_i)$ and consequently $S^2 = \sum_{i=1}^{n} d_i^2 \ge 0$. Only if the assumed values of parameters are the true values, S^2 can be zero, but smaller it is, closer are the assumed values of the parameters from their true values (assuming empirical uniqueness of the parameters to a given set of data). Thus we have to choose the values of Gielis parameters in $g(.)$ such that S^2 is minimum.

Minimization of S^2 poses formidable problems due to two reasons. First, the Gielis parameters are possibly not unique to data. The trio of n_1, n_2 and n_3 is possibly not independent of each other. In econometric literature, the multicollinearity problem presents an instance of a lack of such uniqueness. This fact apart, the parameters span a highly nonlinear surface of S^2, which has innumerably many local minima. The author has observed that a highly adaptive direct search method such as the Nelder-Mead algorithm (Nelder and Mead, 1964; Kuester and Mize, 1973) is often caught into local optima while finding the optimal values of the Gielis parameters. Multiple starting points sometimes succeed, but often fail to give a desirable result.

The Genetic Algorithm for Optimization: While faced with the problem of optimization of a multi-parameter nonlinear function with innumerably many local optima, a choice of the genetic algorithm to find the best possible global optimum is perhaps appropriate. With this point, we have applied the genetic algorithm (Holland, 1975; Goldberg, 1989; Write, 1991) to fit the Gielis superformula to the data experimentally generated by simulation.

The Simulation Experiments: We have experimented with six different models. The parameters by which data have been generated are presented in tables A.1 and A.2 in the appendix. The graphical view of the shapes may be appreciated by looking at the Fig.1 through Fig.18. The red points are those generated by the true parameters while the blue ones are generated by using the estimated parameters.

For every model (1 through 6) three estimates have been obtained. The considerations for the alternative estimates are as follows:

1. By observing the graph plot of the data (100 points for each model) generated by the true parameters, we may guess on the ranges of the first four parameters, c_x, c_y, a, b rather easily.
2. The value of m may be guessed more or less correctly, within a narrow interval, by observing the graph plot.
3. The values of exponential parameters, n_1, n_2, n_3 are hard to guess. However, multi-petal shapes such as the graphs of models 1, 2 and 5 suggest that n_2 or n_3 may be negative.
4. On this basis we have made a three-fold classification: (a) **Most Informed** – when the signs of exponential parameters are known and their range is narrow; (b) **Less Informed** – when the sign of the exponential parameters are known but the range is quite wide; and (c) **Least Informed** – when the signs of the parameters are not known and the range within which they may lie is quite wide.

Accordingly, we have arbitrarily assumed the ranges in case of each parameter and searched for the least value of S^2. In each case, we have run the genetic algorithm program (in FORTRAN) 30 times. It may be noted that the genetic algorithm does have a great capability of adaptation, but it may also be caught in a local optimum. After all, it is a random search algorithm, although much better algorithm than the usual ones. The least value of S^2 in 30 runs is chosen as the best point, nearest to the optimum. Of course, 100 runs might have produced still better results. But our purpose in this paper is not to obtain the best results; we aim at showing that the Gielis curves may be fitted to data successfully by using the genetic algorithms.

The Findings: The details of the findings (estimated parameters, assumed ranges, extent of fitness or S^2, etc) are presented in tables A.1 and A.2 in the appendix. The points generated by true parameters are red-coloured. For each case, 1000 points are generated by the estimated parameters. These points are blue-coloured. A visual observation suggests that in most cases the true shapes have been satisfactorily reproduced by the estimated parameters.

Table-1 presents the fitness in view of the information used in determining the ranges in which the parameters are likely to lie. We observe that most informed guesses on ranges produce better fits.

Table-1. Level of Information about the Rage of Parameters on Extent of Fitness

Information Level	Model-1	Model-2	Model-3	Model-4	Model-5	Model-6
Most Informed	0.347	0.004	0.015	0.016	0.025	0.000
Less Informed	0.618	0.006	0.025	0.105	0.080	0.000
Least Informed	3.640	0.015	0.027	0.114	0.496	0.006

Figures in the table are S^2 indicating the extent of fit. For details see tables A.1 and A.2 in the Appendix.

A perusal of the magnitudes of exponential parameters (n_1, n_2, n_3) suggests that very often they deviate significantly from the true parameters but they may have a tendency to keep some proportionate relations among themselves. This tendency indicates the lack of empirical uniqueness of the parameters of the Gielis superformula (Mishra, 2006). For model-6 in particular, the three estimates give vary small and practically indistinguishable S^2, but variations in the estimated parameters are so large. On the basis of fitness (alone) one cannot say whether the one estimate (of parameters) is more reliable than the others. Thus, the exponential parameters intermingle with each other. However, this is only a conjecture of the author, albeit supported by enough experimental evidences. This issue deserves further investigation. A binomial expansion of $\left(u^{\alpha}+v^{\beta}\right)^{\gamma}$ may indicate how the exponents interact among themselves leading to their empirical indeterminacy.

Modification of Deformed Ellipse by Other Functions: The deformed ellipse ($g(\theta)$ in eqn. (4)) may be modified by another function, $f(\theta)$. Such a modification may alter the shape of either function in a very interesting manner. For example, a triangular shape of model-4 if modified by $f(\theta)=|\cos(w\theta)|$ may give a shape as in model-12 (see Fig-24 in the appendix).

We have conducted some experiments with such modifications of the deformed ellipses. Models 7 , 8 and 9 are the modified versions of models 2, 3 and 5 respectively. The modifier is the logarithmic spiral function, $f(\theta)=n_4.\exp(n_5\theta)$; $n_4=1.1, n_5=0.1$. The estimated parameters and other details are given in table A.3. Models 10 and 11 are instances of two different types of $g(\theta)$ modified by $f(\theta)=n_4.|\cos(n_5\theta)|$; $n_4=1, n_5=2.5$. Lastly, the model-12 is the modified model-5 where the modifier is $f(\theta) = n_4.|\cos(n_5\theta)|$; $n_4 = 1, n_5 = 2.5$. The estimated parameters and other details are given in table A.3.

A variety of figures are deformed ellipses or triangles modified by a cardioid or limaçon of Pascal, i.e. $r = a \pm b\cos(\theta)$. Such examples are numerous in the plant kingdom (leaves in particular). Model-13 (a and b) simulates such cases. Two simulation cases with different limits are tried (table A-3). The figures are presented in Fig-25 and Fig-26. In model-14 measurements (21 in number) are taken from a (real-life) rose leaf. The estimated parameters are presented in table A.3 and the figure Fig.27.

The results of experiments with modification suggest that while the parameters of the modifier functions, $f(\theta)$, are more or less close to the true parameters, the exponential parameters of $g(\theta)$ vary so widely.

Interpretation of Gielis Parameters: Gielis has not given any convincing arguments to explain the 'physical' meaning of the parameters of his equation. Nevertheless, an instance from the econometric literature may be useful to work out a plausible interpretation.

Arrow et al. (1961), later on generalized for multi-level multi-factor case (Sato, 1970), visualized the technical relationship between production (Q) and the factors (Labour, L and capital, K) in the following form (well known as the CES production function):

$$Q = A[\delta L^{-\beta} + (1-\delta)K^{-\beta}]^{-\eta/\beta} ;\; 0 < \delta < 1;\; \beta \geq -1;\; \eta > 0;\; A > 0. \qquad \ldots (7)$$

Writing $x = L^{-1}$ and $y = K^{-1}$ we may rewrite (7) in a form that resembles the Galis superformula (4). The points of difference, nevertheless, are that Gielis (i) does not use the distribution parameter (δ), (ii) does not restrict the substitution parameter (β), (iii) allows for different exponents on x and y, and finally, (iv) allows for the rotational symmetry with the parameter m other than 4 as well, while (7) assumes $m = 4$ tacitly.

In the function (7) above, the substitution parameter, β, measures the technical possibility of substitution between labour and capital without affecting the quantity of production, Q. From this we get the so-called elasticity of substitution, $\sigma = 1/(1+\beta)$. When $\sigma \simeq 0$ we obtain the Leontief type of production function in which the ratio of labour to capital is fixed and they cannot be substituted for each other. On the other hand, when $\sigma = 1$, we have the Cobb-Douglas production function and so on. The homogeneity parameter, η, measures the returns to scale. If the quantities of labour and capital (both) are multiplied by a factor $(1+\lambda)$ then as a response, Q is multiplied by a factor $(1+\lambda)^{\eta}$. The scale economies (or diseconomies) are generated by the movement of the factor proportions towards (or backwards) the optimal mix of inputs.

In case of a leaf we find it producing many types of output for the use by a plant - for preparing food, respiration, balance of water, reproduction, and so on. On the other hand, venation greatly determines the dimensions of a leaf and its functions. In the light of substitutability, complementarity and scale economies one may venture upon some plausible interpretation of the Gielis parameters.

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Appendix

Table-A.1. True and Estimated Gielis Parameters with the Limits on them for Estimation

Parameters		C_x	C_y	a	b	n_1	n_2	n_3	m	S^2
MODEL #1	Tru	0.0	0.0	1.0	1.0	2.0	-3.0	-2.0	10.0	0.0
	LL	-1.0	-1.0	0.9	0.9	0.0	-10.0	-10.0	7.0	Most
	UL	1.0	1.0	1.1	1.1	10.0	-0.01	-0.01	12.0	INF
	EP	0.0006	0.0001	1.0373	1.0637	3.8576	-6.3656	-4.0121	10.0766	0.3469
	LL	-1.0	-1.0	0.9	0.9	0.0	-40.0	-40.0	7.0	Less
	UL	1.0	1.0	1.1	1.1	40.0	-0.01	-0.01	12.0	INF
	EP	-0.0048	-0.0004	1.0677	1.0863	11.2376	-16.8018	-14.1361	9.9669	0.6184
	LL	-1.0	-1.0	0.9	0.9	-20	-40.0	-40.0	7.0	Least
	UL	1.0	1.0	1.1	1.1	20.0	40.0	40.0	12.0	INF
	EP	-0.0532	0.0330	1.0712	1.0910	17.9964	-30.1576	-23.4136	10.9931	3.6401
MODEL #2	Tru	0.0	0.0	1.0	1.0	8.0	4.0	-4.0	6.0	0.0
	LL	-1.0	-1.0	0.9	0.9	0.0	0.0	-20.0	5.0	Most
	UL	1.0	1.0	1.1	1.1	20.0	20.0	-0.10	8.0	INF
	EP	0.0011	-0.0010	0.9574	0.9961	13.0300	2.0972	-6.4395	6.0083	0.0036
	LL	-1.0	-1.0	0.9	0.9	0.0	0.0	-50.0	5.0	Less
	UL	1.0	1.0	1.1	1.1	50.0	50.0	-0.10	8.0	INF
	EP	-0.0042	-0.0019	1.0906	1.0084	14.7780	48.0280	-7.4458	5.9906	0.0058
	LL	-1.0	-1.0	0.9	0.9	-20.0	-30.0	-30.0	5.0	Least
	UL	1.0	1.0	1.1	1.1	20.0	30.0	30.0	8.0	INF
	EP	-0.0000	-0.0026	0.9601	1.0129	12.8016	27.3690	-6.2526	5.9901	0.0153
MODEL #3	Tru	0.0	0.0	1.0	1.0	-3.0	4.0	4.0	12.0	0.0
	LL	-1.0	-1.0	0.9	0.9	-20.0	0.0	0.0	10.0	Most
	UL	1.0	1.0	1.1	1.1	-0.10	20.0	20.0	14.0	INF
	EP	0.0003	-0.0041	1.0100	1.0323	-7.5329	8.2108	5.5700	12.0122	0.0149
	LL	-1.0	-1.0	0.9	0.9	-60.0	0.0	0.0	11.0	Less
	UL	1.0	1.0	1.1	1.1	-0.1	70.0	70.0	16.0	INF
	EP	0.0033	-0.0063	1.0437	1.0038	-9.7169	6.7606	10.3439	12.0104	0.0246
	LL	-1.0	-1.0	0.9	0.9	-50.0	-50.0	-50.0	11.0	Least
	UL	1.0	1.0	1.1	1.1	50.0	50.0	50.0	14.0	INF
	EP	0.0009	-0.0004	1.0118	1.0459	-47.4170	42.1720	31.8440	12.0113	0.0266
MODEL #4	Tru	0.0	0.0	1.0	1.0	0.6	2.0	3.0	3.0	0.0
	LL	-1.0	-1.0	0.9	0.9	0.0	0.0	0.0	1.0	Most
	UL	1.0	1.0	1.1	1.1	7.0	7.0	7.0	7.0	INF
	EP	0.0003	0.0046	1.0083	1.0094	2.6094	5.7157	2.8543	2.9785	0.0160
	LL	-1.0	-1.0	0.9	0.9	0.0	0.0	0.0	2.0	Less
	UL	1.0	1.0	1.1	1.1	80.0	80.0	80.0	9.0	INF
	EP	-0.0008	-0.0029	1.0467	1.0641	62.5760	63.4184	43.6816	3.003	0.1051
	LL	-1.0	-1.0	0.9	0.9	-50.0	-50.0	-50.0	1.0	Least
	UL	1.0	1.0	1.1	1.1	50.0	50.0	50.0	7.0	INF
	EP	0.0010	-0.0022	1.0110	1.0931	29.6980	40.6590	17.7510	3.0039	0.1143

Tru = True Parameters, LL=Lower Limits, UL =Upper Limits, EP = Estimated Parameters

Table-A.2. True and Estimated Gielis Parameters with the Limits on them for Estimation										
Parameters		C_x	C_y	a	b	n_1	n_2	n_3	m	S^2
MODEL # 5	Tru	0.0	0.0	1.0	1.0	8.0	4.0	-4.0	30.0	0.0
	LL	-1.0	-1.0	0.9	0.9	0.0	0.0	-8.0	20.0	Most INF
	UL	1.0	1.0	1.1	1.1	10.0	8.0	-0.1	38.0	
	EP	0.0003	0.0001	1.0571	1.0056	5.5917	3.9574	-2.8744	30.0210	0.0254
	LL	-1.0	-1.0	0.9	0.9	0.0	0.0	-70.0	28.0	Less INF
	UL	1.0	1.0	1.1	1.1	40.0	80.0	-0.1	32.0	
	EP	0.0000	-0.0005	1.0974	0.9923	28.2792	53.5144	-15.9981	30.0234	0.0804
	LL	-1.0	-1.0	0.9	0.9	-70.0	-70.0	-70.0	27.0	Least INF
	UL	1.0	1.0	1.1	1.1	70.0	70.0	50.0	32.0	
	EP	0.0029	-0.0020	0.9714	1.0958	28.7322	27.4134	-13.0928	30.1057	0.4957
MODEL # 6	Tru	0.0	0.0	1.0	1.0	12.0	3.0	-7.0	12.0	0.0
	LL	-0.01	-0.01	0.99	0.99	0.0	0.0	-20.0	11.0	Most INF
	UL	0.01	0.01	1.01	1.01	20.0	20.0	0.0	14.0	
	EP	-0.0000	-0.0000	1.0092	1.0017	14.5776	17.8710	-8.5150	11.9986	0.0003
	LL	-0.01	-0.01	0.99	0.99	0.0	0.0	-80.0	11.0	Less INF
	UL	0.01	0.01	1.01	1.01	80.0	80.0	0.0	14.0	
	EP	0.0000	-0.0000	1.0007	1.0043	23.2632	9.2800	-13.5368	12.0005	0.0002
	LL	-0.01	-0.01	0.99	0.99	-80.0	-80.0	-80.0	11.0	Least INF
	UL	0.01	0.01	1.01	1.01	80.0	80.0	80.0	14.0	
	EP	0.0006	0.0004	1.0029	0.9968	24.4480	33.5680	-14.5504	11.9997	0.0062
Tru = True Parameters, LL=Lower Limits, UL =Upper Limits, EP = Estimated Parameters										

Figures

Fig-1 (Model-1)

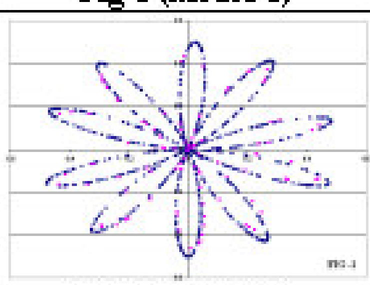


Fig-2 (Model-1)

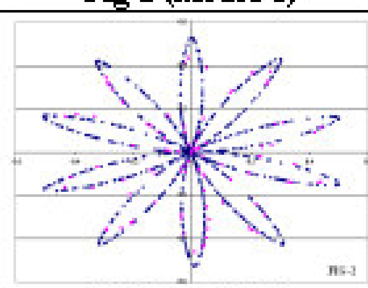


Fig-3 (Model-1)

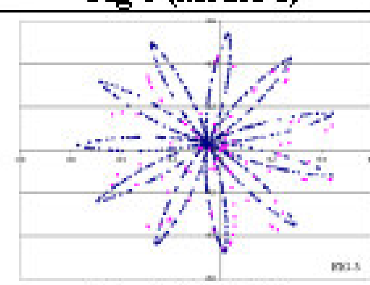


Fig-4 (Model-2)

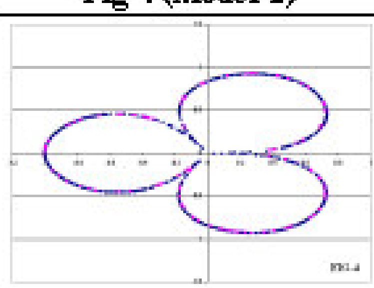


Fig-5 (Model-2)

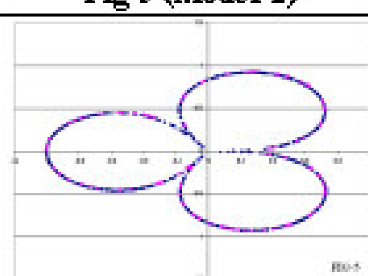


Fig-6 (Model-2)

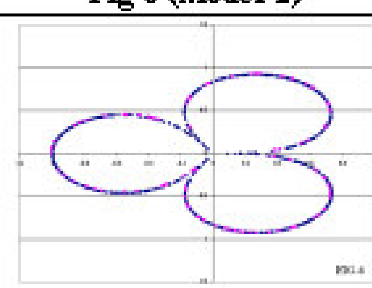


Fig-7 (Model-3)

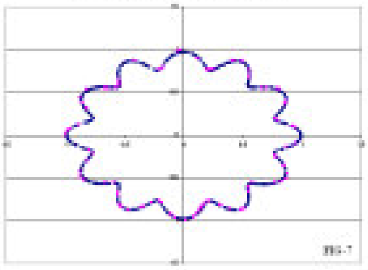


Fig-8 (Model-3)

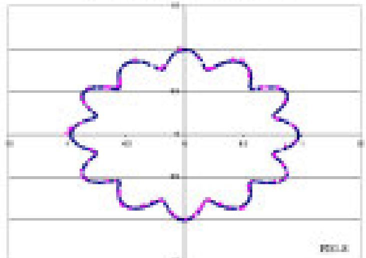


Fig-9 (Model-3)

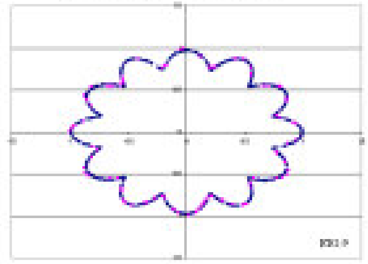


Fig-10 (Model-4)

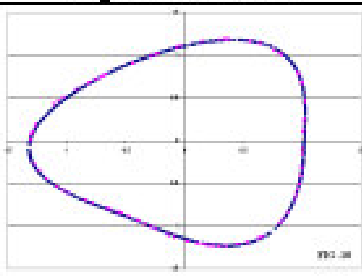


Fig-11 (Model-4)

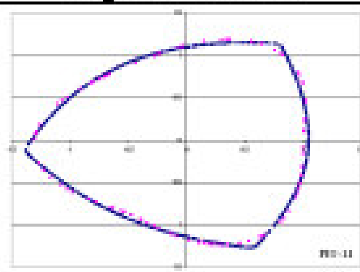


Fig-12 (Model-4)

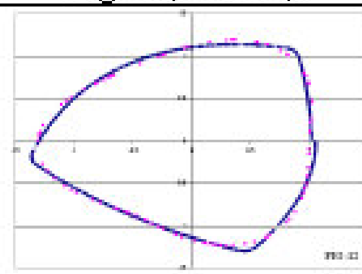


Fig-13 (Model-5)

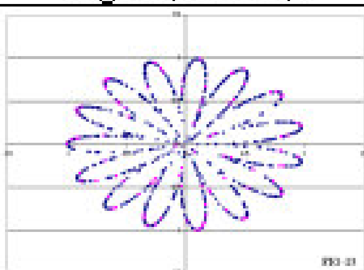


Fig-14 (Model-5)

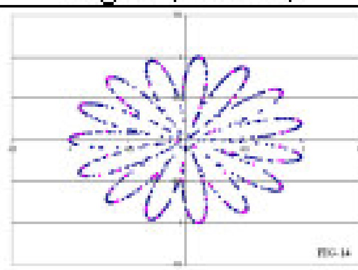


Fig-15 (Model-5)

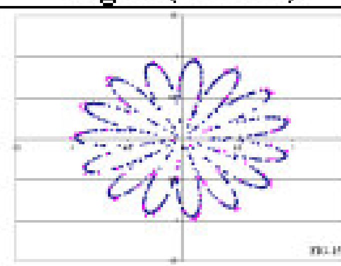


Fig-16 (Model-6)

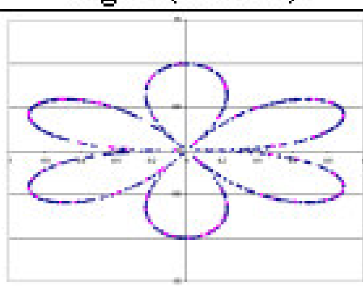


Fig-17 (Model-6)

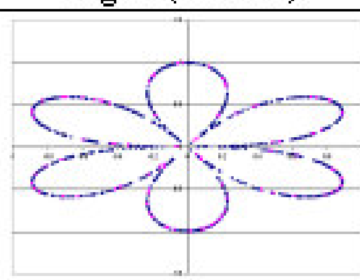


Fig-18 (Model-6)

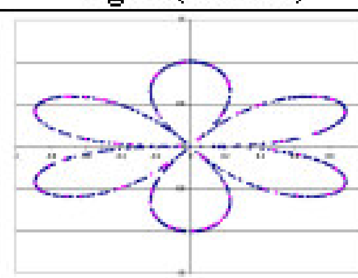


Table-A.3. True & Estimated Gielis Parameters (Modified Curves) with Limits on them												
	c_x	c_y	a	b	n_1	n_2	n_3	m	n_4	n_5	S^2	
M	0	0	1	1	8	4	-4	6	1.1	0.1	0	T
#	-0.1	-0.1	0.9	0.9	-80	-80	-80	2	1.0	0.01	LL	
7	0.1	0.1	1.1	1.1	80	80	80	10	1.5	0.30	UL	
	0.0032	-0.0022	1.0621	0.9188	49.8240	-28.5680	6.0050	57.2304	1.1121	0.0872	0.0933	E
M	0	0	1	1	-3	4	4	12	1.1	0.1	0	T
#	-0.1	-0.1	0.9	0.9	-80	-80	-80	3	1.0	0.01	LL	
8	0.1	0.1	1.1	1.1	80	80	80	20	1.5	0.30	UL	
	0.0022	0.0003	1.0374	1.0958	-67.1024	75.4592	45.6800	11.9911	1.1554	0.0975	0.0961	E
M	0	0	1	1	8	4	- 4	30	1.1	0.1	0	T
#	-0.1	-0.1	0.9	0.9	-80	-80	-80	10	1.0	0.01	LL	
9	0.1	0.1	1.1	1.1	80	80	80	40	1.5	0.30	UL	
	0.0089	-0.0002	0.9236	0.9796	37.0944	67.7456	-15.3184	30.0064	1.0923	0.0920	0.3169	E
M	0	0	1	1	-1.3	2.7	2.7	2.5	1	2.5	0	T
#	-0.1	-0.1	0.9	0.9	-80	-80	-80	2	0.999	2	LL	
10	0.1	0.1	1.1	1.1	80	80	80	3	1.001	3	UL	
	0.0000	0.0001	1.0664	1.0103	37.4576	48.3504	2.4818	-66.1872	1.0010	2.5056	0.0433	E
M	0	0	1	1	2.5	5	5	2.5	1	2.5	0	T
#	-0.001	-0.001	0.999	0.999	-80	-80	-80	2	0.9999	2	LL	
11	0.001	0.001	1.001	1.001	80	80	80	3	1.0001	3	UL	
	-0.0004	0.0010	1.0008	1.0008	22.4576	34.4464	30.072	2.4994	1.0000	2.4998	0.0645	E
M	0	0	1	1	0.6	2	3	3	1	2.5	0	T
#	-0.1	-0.1	0.9	0.9	-80	-80	-80	2	0.999	2	LL	
12	0.1	0.1	1.1	1.1	80	80	80	7	1.001	3	UL	
	0.0000	0.0001	1.0823	1.0453	59.1584	49.6272	46.7280	3.0038	0.9999	2.5021	0.0629	E
M	0	0	1	1	0.6	2	3	3	2	1	0	T
#	-1	-1	0	0	-80	-80	-80	1	1	0.999	LL	
13	1	1	2	2	80	80	80	30	3	1.001	UL	
(a)	-.0288	0.0085	0.9754	1.7805	40.6960	7.6528	16.2752	5.9976	2.1041	0.9996	0.9529	E
M	0	0	1	1	0.6	2	3	3	2	1	0	T
#	-1	-1	0	0	-80	-80	-80	1	1.9	0.999	LL	
13	1	1	2	2	80	80	80	80	2.1	1.001	UL	
(b)	-0.611	0.0440	1.2842	1.2270	50.1120	9.5152	65.4304	5.9881	1.9947	0.9998	1.9820	E
R	Data points from a real Rose Leaf - True Parameters Unknown											
E	-4	-1	4	3	-80	-80	-80	2.9	1.9	0.999	LL	
A	4	1	7	5	80	80	80	30	2.1	1.001	UL	
L	2.7986	0.0614	4.0662	4.8729	71.1760	51.9328	8.3136	4.0046	1.9935	1.0010	1.9717	E
M=Model; LL=Lower Limits; UL=Upper Limite; T=True Parameters; E=Estimated Parameters												

Fig-19 (Model-7)	Fig-20 (Model-8)	Fig-21 (Model-9)

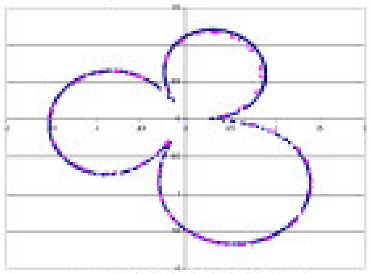
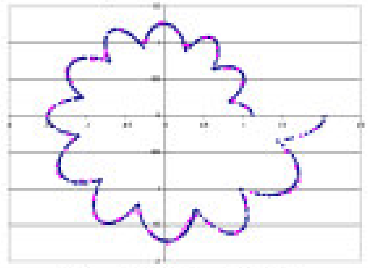
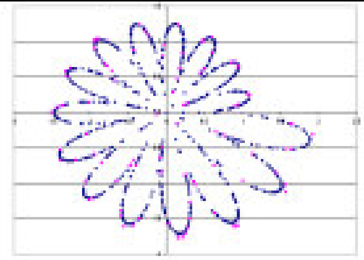

		

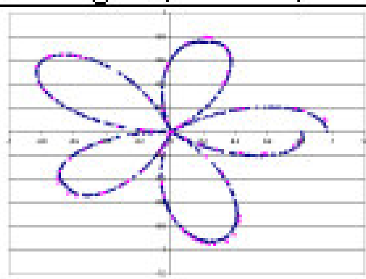
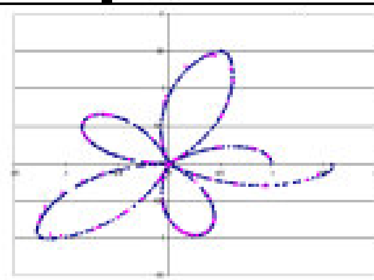
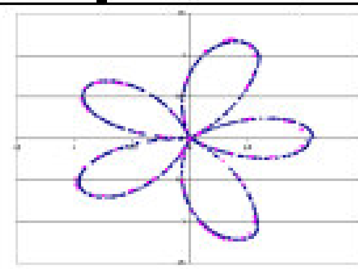
Fig-22 (Model-10)	**Fig-23 (Model-11)**	**Fig-24 (Model-12)**
		

Fig-25 (Model-13(a))	Fig-26 (Model-13(b))	Fig-27 (Real Data)